Problem 1

1. Suppose that f is continuous and that the sequence

$$x, f(x), f(f(x)), f(f(f(x))), \ldots$$

converges to L. Prove that L is a "fixed point" for f, i.e., $f(L) = L$.

2. A function $f : [a, b] \to [a, b]$ is called a contraction if there exists $c < 1$ such that, for all $x, y \in [a, b]$,

$$|f(x) - f(y)| \leq c|x - y|.$$

Prove that any contraction has a unique fixed point.

Recall that if $(s_n), (t_n)$ are sequences of real numbers, and for some $N \in \mathbb{N}$ we have $n > N$ implies $s_n \leq t_n$, then

$$\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n$$
$$\limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n$$

and for any bounded sequence (a_n), $\lim_{n\to\infty} a_n = L$ if and only if

$$\liminf_{n\to\infty} a_n = L = \limsup_{n\to\infty} a_n.$$

Also recall the **binomial theorem**: for all $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Problem 2

1. Prove that $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

2. Prove that for all $n \in \mathbb{N}$,

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right).$$

   Conclude that $\left(1+\frac{1}{n}\right)^n \leq \sum_{k=0}^{n} \frac{1}{k!}$.

3. Prove that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \geq \sum_{n=0}^{\infty} \frac{1}{n!}.$$

   Conclude that $\sum_{n=0}^{\infty} \frac{1}{n!} = e$. *Hint*: Fix some $m \in \mathbb{N}$ and show $\sum_{k=0}^{m} \frac{1}{k!} \leq \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$, then let m go to infinity.

4. Show that for all $n \in \mathbb{N}$,

$$e - \sum_{k=0}^{n} \frac{1}{k!} = \sum_{m=n+1}^{\infty} \frac{1}{m!} < \frac{1}{n!n}.$$

   How many terms of this series do we need to compute e accurate up to 10 decimal places?

5. Prove that e is irrational. *Hint*: If $e = p/q$ for $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, then

$$0 < q!q\left(e - \sum_{k=0}^{q} \frac{1}{k!}\right) < 1.$$

How does this lead to a contradiction?

Problem 3
Show that if $\lim_{n\to\infty} a_n = L$ then

$$\lim_{n\to\infty} \frac{(a_1 + \cdots + a_n)}{n} = L.$$

Hint: Separate and bound: for large enough N, a_N is close to L. This means $a_N + \cdots + a_{N+M}$ is close to $M \cdot L$, so $(a_N + \cdots + a_{N+M})/(N+M)$ is close to $(M \cdot L)/(M+N)$. If M is large in comparison to N, then $(M \cdot L)/(M+N)$ is close to L.